Mixing-layer flows in a saturated porous medium

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The steady two-dimensional seepage flow by gravitational convection, of a fluid of density ρ_1 surrounded by a fluid of density ρ_0 ($\neq \rho_1$) at rest, leads to a potential problem from which the shape of the interface can be determined. When the two fluids are slightly miscible the interface is replaced by a mixing layer, and it is shown that the first-order Prandtl equations for the flow in the layer possess an exact similarity solution. The profile across the layer is of the same form as the profile of the laminar incompressible half-jet with one fluid at rest, and a formula is obtained for the scale of mixing-layer thickness as a function of distance downstream. Three examples are discussed.

(*a*) Flow of fluid of density ρ_1 from a horizontal line source. When $\rho_1 > \rho_0$ a stagnation point exists above the source, and the fluid ultimately descends in a vertical column of width proportional to the source strength. At the stagnation point, the mixing-layer thickness is finite and is proportional to the square root of the radius of curvature of the interface. At a sufficiently great distance downstream, the thickness increases as the square root of the distance, as in the straight laminar half-jet. These results have been tested experimentally in a Hele–Shaw cell.

(*b*) Symmetrical flow of an ascending column of fluid ($\rho_1 < \rho_0$) about an obstacle in the form of a finite horizontal strip. The column reforms after passing the obstacle, and the mixing-layer thickness returns to the value corresponding to an unobstructed vertical half-jet. The flow has been produced experimentally.

(*c*) Flow in a lens of fresh water overlying salt water, with inflow due to precipitation, as in a two-dimensional Ghyben–Herzberg lens. Here the potential-flow solution is calculated approximately by means of Dupuit–Forchheimer theory. In the steady-state solution the thickness of the mixing layer between fresh and saline water is found to be finite and, as in (*a*), proportional to the square root of the radius of curvature of the lens.

1. Introduction

A recent paper (Wooding 1963, subsequently denoted by I) describes steady vertical seepage flows in a saturated porous medium at large Rayleigh or Péclet number when free boundary layers (mixing layers) are present. These flows are governed by equations similar to those of laminar incompressible flow in such cases as the half-jet (Görtler 1942) and the momentum jet from a slit or point source (Schlichting 1933)—a consequence of the linear relationship between fluid velocity and net buoyancy in the equations of motion (Darcy's law).

Since pressure terms are neglected in the boundary-layer equations, the vertical velocity component can be substituted for density in the equation of mass transport, and an equation of quasi-momentum transport results.

An extension of the above principle to a curved mixing layer is slightly more complicated since the layer is not parallel to the buoyancy force; at some points it may be normal to it. However, the steady flow under gravity of a homogeneous fluid through a porous medium containing a second fluid at rest involves no special difficulty. In this case externally applied pressure gradients are absent, and a point at which the slope of the mixing layer is zero corresponds to a stagnation point. The first-order boundary-layer equations are well behaved in the neighbourhood of such a point, and a similarity solution for the flow in the mixing layer is readily found.

In §2, the first-order equations of the mixing layer are derived, and the similarity solution is given in §3. The solution for a stagnation-point flow from a horizontal line source is given in §4, the case of flow past an obstacle in §5, while §6 contains an approximate treatment of the interaction of ground water and sea water in a two-dimensional Ghyben–Herzberg lens.

2. The mixing-layer equations

In defining the co-ordinate systems, it is convenient to refer to figure 1, which illustrates part of a steady plane gravity seepage flow of fluid of density ρ_1 below a fluid of density ρ_0 ($< \rho_1$) at rest. Curvilinear co-ordinates (x, y) are taken with x measured along the 'interface' OR separating the two miscible fluids; the point O is a stagnation point at which the slope is zero, and at R the slope-angle is designated by $\alpha(x)$. Although a true interface does not exist, OR is determined by solving the potential problem which results when mutual diffusivity is neglected. For this problem, the Cartesian co-ordinates (X, Y) are suitable.

It will be assumed that all variables have been rendered dimensionless in terms of parameters defined in relation to the potential flow as follows:

(i) a length L equal to the radius of curvature at the stagnation point O in figure 1,

(ii) a density difference $|\rho_1-\rho_0|$ (usually $|\rho_1-\rho_0|/\rho_0 \ll 1$);

(iii) a flow rate
$$U_m = (gk/\mu)\,|\rho_1-\rho_0|, \tag{1}$$
equal to the rate of vertical gravity flow of a column of fluid of density ρ_1 surrounded by a fluid of density ρ_0 at rest. In this definition, the permeability k is assumed to be a constant, and variations in viscosity μ are assumed negligible.

The dimensionless flow vector will be designated by either (u, v), or (U, V) depending upon the choice of co-ordinate system, and dimensionless pressure and density differences will be defined as
$$p = \frac{P-g\rho_0 LX}{g|\rho_1-\rho_0|\,L}, \quad \theta = \frac{\rho-\rho_0}{|\rho_1-\rho_0|}, \tag{2a, b}$$
where P is the actual pressure and $g\rho_0 LX$ is the hydrostatic head at the same depth in the fluid at rest. The Rayleigh number
$$\lambda = \frac{gk|\rho_1-\rho_0|\,L}{\kappa\mu} \tag{3}$$

is assumed very large. Here the diffusivity κ can be taken isotropic and constant provided that the flow rate is sufficiently low (cf. I).

In terms of the curvilinear co-ordinates (x, y) the equations of continuity, motion and mass transport involve coefficients with factors of magnitude $1+O(y\alpha')$ (Schlichting 1960, p. 111), where the curvature $\alpha' = d\alpha/dx$ does not exceed $O(1)$. Since $y = O(\lambda^{-\frac{1}{2}})$ in the neighbourhood of the mixing layer, the quantity $y\alpha'$ can be neglected relative to unity.

If y and v are replaced by $\lambda^{-\frac{1}{2}}y$ and $\lambda^{-\frac{1}{2}}v$ preparatory to taking a Prandtl limit, the equations of the flow can be written as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = O(\lambda^{-\frac{1}{2}}), \tag{4}$$

$$\frac{\partial p}{\partial x}+u-\theta\sin\alpha = O(\lambda^{-\frac{1}{2}}), \tag{5}$$

$$\lambda^{\frac{1}{2}}\frac{\partial p}{\partial y}+\lambda^{-\frac{1}{2}}v+\theta\cos\alpha = 0, \tag{6}$$

$$u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}-\frac{\partial^2\theta}{\partial y^2}-\lambda^{-1}\frac{\partial^2\theta}{\partial x^2} = O(\lambda^{-\frac{1}{2}}), \tag{7}$$

where the contributions due to the use of curvilinear co-ordinates have been placed on the right-hand sides. Now, from the second equation of motion (6) it is evident that the total change in p across the mixing layer is of order $\lambda^{-\frac{1}{2}}\,\theta\cos\alpha$ and, since $p = 0$ in the fluid at rest outside the layer, equation (5) gives

$$u = \theta\{\sin\alpha+O(\lambda^{-\frac{1}{2}}\cos\alpha)\}. \tag{8}$$

After θ has been eliminated between (7) and (8), letting $\lambda\to\infty$ leads to

$$\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}-\frac{\partial^2}{\partial y^2}\right)\left(\frac{u}{\sin\alpha}\right) = 0. \tag{9}$$

The first-order Prandtl equations (9) and (4) are to be solved subject to the boundary conditions $\theta(x,\infty) = 0$ and $\theta(x,-\infty) = 1$, i.e. from (8),

$$u(x,\infty) = 0, \quad u(x,-\infty) = \sin\alpha(x). \tag{10a, b}$$

3. The similarity solution

Let the stream-function ψ defined from (4) in the usual way be written as

$$\psi = h^{\frac{1}{2}}(x)f(\eta), \tag{11a}$$

where

$$\eta = yh^{-\frac{1}{2}}(x)\sin\alpha(x). \tag{11b}$$

Then

$$u = \partial\psi/\partial y = f'\sin\alpha, \tag{12a}$$

$$v = -\partial\psi/\partial x = \tfrac{1}{2}h^{-\frac{1}{2}}h'(\eta f'-f)-h^{\frac{1}{2}}\alpha'\cot\alpha\,\eta f', \tag{12b}$$

and equation (9) reduces to $h'ff''+2f'''\sin\alpha = 0$. The mixing layer is self-similar if

$$h'(x) = \sin\alpha(x), \tag{13}$$

and the differential equation becomes

$$ff''+2f''' = 0, \tag{14}$$

which is the well-known profile equation for the laminar incompressible half-jet of Görtler (1942). In the present case the boundary conditions are $f'(\infty) = 0$, $f'(-\infty) = 1$, corresponding to the half-jet when one fluid is at rest. Solutions have been given by Chapman (1949), Lessen (1949), Lock (1951) and Crane (1957).

From (11*b*) and (13), the mixing-layer thickness varies with x as

$$h^{\frac{1}{2}}/\sin\alpha = \left\{\int_{x_0}^{x} \sin\alpha(x)\,dx\right\}^{\frac{1}{2}} \Big/ \sin\alpha(x), \tag{15}$$

where x_0 is a constant. The right-hand side of (15) is specified from the equation $\alpha = \alpha(x)$, which is the intrinsic equation of the interface between the two fluids.

In terms of the dimensionless Cartesian co-ordinates (X, Y), equation (15) becomes

$$\frac{h^{\frac{1}{2}}}{\sin\alpha} = \frac{(X-X_0)^{\frac{1}{2}}}{\sin\alpha(X)}, \tag{16}$$

where the constant X_0 can be interpreted as the X co-ordinate of an ideal source equivalent to the actual source. Equation (16) reveals that h is a function of $X - X_0$ alone and that, at any point where the interface between the two fluids is vertical, the mixing-layer thickness should be equal to $(X - X_0)^{\frac{1}{2}}$ corresponding to a vertical Görtler type of flow. Clearly, this property is a consequence of the similarity assumption (13) and may be made the basis of an experimental check of its validity.

A particular illustration of the above effect is noted in flow past an obstacle (see §5). Here the mixing-layer thickness downstream of the obstacle apparently tends to the value which would hold if the obstacle were not present—a result of interest in geothermal applications.

The remaining sections, §§4, 5 and 6 deal with applications of the above equations to specific cases.

4. Stagnation flow from a horizontal line source

In figure 1, suppose that the line source S is of strength Q per unit length and that the porous medium is effectively of unlimited extent.

Let $Z = X + iY$, and let $w = \phi + i\psi$ be a dimensionless complex potential of scale-unit $U_m L$ (see (i) and (iii) of §2); the components of the flow-rate are $U = \partial\phi/\partial X = \partial\psi/\partial Y$ and $V = \partial\phi/\partial Y = -\partial\psi/\partial X$. Then the equation of motion (Darcy's law) gives

$$p + \phi = X, \tag{17}$$

where p is defined in (2*a*). The boundary conditions follow from symmetry and the fact that OR (in figure 1) is a streamline along which $p = 0$, and the potential-flow solution for the line source is found to be

$$w = \tfrac{1}{2}\log\{1-(e^Z-1)^2\}$$

(Polubarinova-Kochina 1963). The stagnation point O occurs at a dimensionless distance of log 2 above S, and the fluid ultimately descends in a two-dimensional column of dimensionless width $Q/U_m L = \pi$. For the interface OR the equation is $X = -\log(\cos Y)$, giving, for the intrinsic equation,

$$x = \log\tan(\tfrac{1}{2}\alpha + \tfrac{1}{4}\pi). \tag{18}$$

From (15) and (18), the mixing-layer thickness varies with x as

$$h^{\frac{1}{2}}/\sin\alpha = (\log\cosh x)^{\frac{1}{2}}/\tanh x, \tag{19}$$

the constant x_0 being zero since x is measured from O.

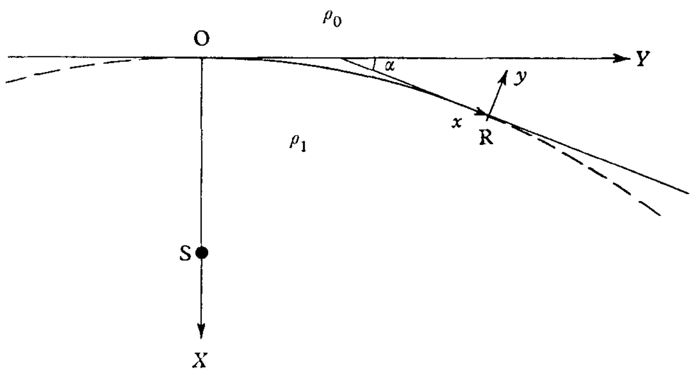

FIGURE 1. Plane symmetrical stagnation flow from a horizontal line source S for the case $\rho_1 > \rho_0$.

In the neighbourhood of the stagnation point at O, the intrinsic equation (18) reduces to $\alpha = x + O(x^3)$, while (19) becomes

$$h^{\frac{1}{2}}/\sin\alpha = 2^{-\frac{1}{2}} + O(x^2). \tag{20}$$

Thus the mixing-layer thickness tends to a finite limit which, in dimensional units, is proportional to $L\lambda^{-\frac{1}{2}}$, i.e. to the square root of the radius of curvature at the stagnation point. It may be noted that, if α were of order x^s as $x \to 0$, the mixing-layer thickness would be of order $x^{\frac{1}{2}(1-s)}$. This corresponds to a finite limit only when $s = 1$.

When $x \to \infty$, (19) gives

$$h^{\frac{1}{2}}/\sin\alpha = (x - \log 2)^{\frac{1}{2}}\{1 + O(e^{-2x})\} \sim X^{\frac{1}{2}}, \tag{21}$$

as would be expected from (16). This is equivalent to a vertical Görtler half-jet from an ideal source located at the same level as the stagnation point.

Flow from a two-dimensional source has been reproduced experimentally in a Hele–Shaw cell, as described in I and illustrated in figures 2 and 5 of that paper. The cell is initially filled with water, and the source fluid is a dilute solution of potassium permanganate which forms a descending column, or jet, of finite width.

Relative values of mixing-layer thickness with distance down-stream have been obtained from photographic-density measurements. In figure 2, these values are superimposed upon a curve calculated from (19), and indicate quite good agreement. The deduction that the mixing layer is of finite thickness at the stagnation point appears to be confirmed.

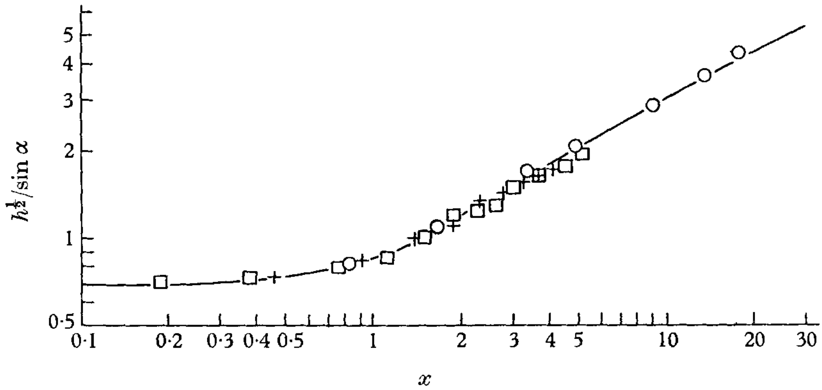

FIGURE 2. Mixing-layer thickness $h^{\frac{1}{2}}/\sin\alpha$ for flow near a stagnation point. ———, Calculated from expression (19). Experimental measurements: ○, $\lambda = 200$; +, $\lambda = 500$; □, $\lambda = 1000$.

5. Flow about a two-dimensional obstacle

A situation of interest in a geothermal application (cf. I) arises when sheets of impermeable solidified lava are encountered in the path of moving heated ground water. The existence of finite horizontal barriers of this type has been established in the geothermal area of Wairakei, New Zealand.

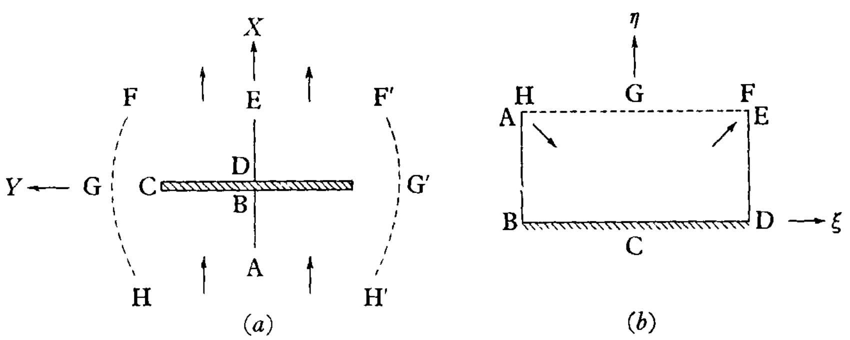

FIGURE 3. Potential-flow problem for a horizontal strip placed symmetrically in the path of a fluid column ascending under gravity. (a) The physical Z-plane; (b) the auxiliary ζ-plane.

Figure 3(a) shows a symmetrical two-dimensional gravity flow about a horizontal strip CC′ for the case when the less dense fluid is in motion. Here AE is the axis of symmetry while FGH and F′G′H′ indicate the positions of the mixing layers. It is assumed that a source of strength Q is located at a sufficiently great distance, in the direction of negative X, for source effects to be negligible. The various dimensionless quantities will be defined as in §4.

Superficially, there appears to be a lack of uniqueness in the solution for the steady potential flow, since the fluid passing the obstacle might either (i) remain in two distinct ascending columns, or (ii) recombine to form a single-column flow of the same asymptotic width as before. However, in the presence of diffusion only (ii) is possible in the steady state, as stagnant fluid of a different density in the lee of the obstacle is gradually removed by entrainment into the moving fluid. Figure 4(*a*) to (*c*), plate 1, shows three stages of this process in a Hele–Shaw cell.

For the solution of the potential problem, it is convenient to take the region ABCDEFGH in figure 3(*a*), and to define an equivalent rectangular region in an auxiliary ζ-plane (figure 3*b*), putting $\zeta = \xi + i\eta$. The Zhukovsky potential is defined as

$$W = \Phi + i\Psi = w - Z$$
$$= (\phi - X) + i(\psi - Y), \qquad (22)$$

and the solution for the potential flow can be written in the parametric form

$$w = w(\zeta), \quad Z = w(\zeta) - W(\zeta). \qquad (23)$$

For the boundary conditions, $\psi = 0$ and $\Psi = 0$ ($Y = 0$) on AB and DE, $\psi = 0$ and $\Phi = \phi$ ($X = 0$) on BCD, and $\psi = \frac{1}{2}\pi$ and $\Phi = 0$ ($\phi = X$) on FGH.

The equation

$$\tanh w = \gamma \operatorname{sn}(2K\zeta/\pi, \gamma) \qquad (24)$$

maps w on to ζ provided that the dimensions of the rectangular region satisfy $\xi = \pm\frac{1}{2}\pi$ on DE and AB and $\eta = \frac{1}{2}\pi K'/K$ on FGH. In (24) the notation follows that of Whittaker & Watson (1950, ch. 22) with the symbol k replaced by γ; sn signifies a Jacobian elliptic function and $K(\gamma)$ the complete elliptic integral of the first kind, while $K' \equiv K(\gamma')$ where $\gamma'^2 = 1 - \gamma^2$.

A convenient representation of W is the Fourier series

$$W = -i \sum_{n=0}^{\infty} (-)^n \frac{\cos(2n+1)(\zeta - \frac{1}{2}i\pi K'/K)}{(n+\frac{1}{2})\sinh^2(n+\frac{1}{2})\pi K'/K}, \qquad (25)$$

valid in the strip $-\frac{1}{2}\pi K'/K < \eta < \frac{3}{2}\pi K'/K$ which includes the region of interest. It is easily shown that (25) satisfies the condition $\Phi = \phi$ on BCD by expanding w in a Fourier series valid in the strip $|\eta| < \frac{1}{2}\pi K'/K$.

Upon the interface FGH, $\zeta - \frac{1}{2}i\pi K'/K = \xi$ in (25) and, if $u = 2K\xi/\pi$, an appropriate expansion of (24) is

$$w = \tanh^{-1}(\operatorname{ns} u) = \int \operatorname{dc} u \, du$$
$$= \log\tan(\tfrac{1}{2}\xi + \tfrac{1}{4}\pi) + \sum_{n=0}^{\infty} (-)^n \frac{\exp\{-(n+\frac{1}{2})\pi K'/K\}\sin(2n+1)\xi}{(n+\frac{1}{2})\sinh(n+\frac{1}{2})\pi K'/K} + i\tfrac{1}{2}\pi. \qquad (26)$$

The (X, Y) co-ordinates then follow from (23), while the point C$(0, a)$ is given by the formula $a = -\operatorname{Im}\{W(0)\}$.

Figure 5(*a*) has been plotted for the case $a = 2{\cdot}5$, and also shows experimental measurements of the position of the flow interface illustrated in figure 4(*c*).

For the mixing layer, the formulae (23), (25) and (26) serve to evaluate X and $\sin\alpha = \{1 + (Y'/X')^2\}^{-\frac{1}{2}}$ in (16). The thickness scale $h^{\frac{1}{2}}/\sin\alpha$ has been plotted in figure 5(*b*) as a function of X for the parameter values $a = 2{\cdot}5$, $X_0 = 10$, corresponding approximately to the experimental case illustrated in figure 4(*c*) where the mixing-layer thickness was measured on an arbitrary scale. There is quite good agreement with the experimental values; in particular, the function

$h^{\frac{1}{2}}/\sin\alpha$ is observed to return approximately to the Görtler value of $(X-X_0)^{\frac{1}{2}}$ after passing the obstacle.

A peculiar feature of figure 4(c), which should be mentioned, is the appearance of a thin dark line, surrounding the main flow, after steady-state conditions have been established. This arises through the very gradual oxidation of organic

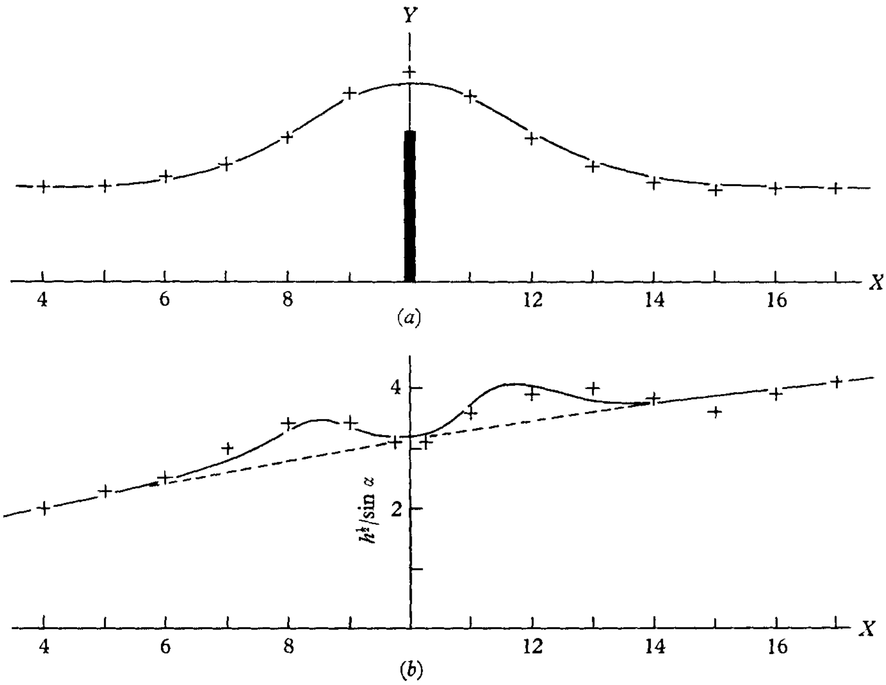

FIGURE 5. (a) One-half of a symmetric potential flow about a finite horizontal strip. ———, From equations (23), (25) and (26) for $a = 2{\cdot}5$; +, experimental measurements from the flow illustrated in figure 4(c) where $a = 2{\cdot}65$. (b) Mixing-layer thickness $h^{\frac{1}{2}}/\sin\alpha$ for flow about a horizontal strip. ———, From equations (16), (23), etc., for $a = 2{\cdot}5$; – – –, from the formula $h^{\frac{1}{2}}/\sin\alpha = X^{\frac{1}{2}}$; +, experimental measurements from the flow illustrated in figure 4(c).

impurity in the entrained water. Precipitation of manganese dioxide begins when the solubility product (a very small number) is exceeded, which can be expected to occur where the concentration of source fluid has attained a certain critical value. The distance of the precipitate zone from the main flow has been found to be roughly proportional to mixing-layer thickness.

6. The Ghyben–Herzberg lens

In the interaction of ground water and sea water at a coast, it is frequently observed that a wedge of fresh water overlies saline water inside the permeable soil. A particularly well-known example, found underground in some oceanic islands, is the Ghyben–Herzberg lens (Wentworth 1947; Childs 1950; Carrier 1958). Such a system appears capable of maintaining itself indefinitely, and it is of interest to consider whether the steady-state mixing-layer theory can represent the main features of the flow. This involves the neglect of tidal movement.

However, if the Rayleigh number, (3), is large when κ is replaced by an appropriate 'effective diffusivity' (which includes the effect of tidal dispersion), the treatment of §§2, 3 should remain valid.

Figure 6 shows a symmetrical two-dimensional flow of fresh water of density ρ_1, with stagnation point at C, overlying saline water of density ρ_0 ($>\rho_1$) at rest. The Y-axis is taken at sea-level, with coast-lines at B and D. It is convenient here to adopt a length-scale L_1 equal to OD, but to retain U_m (defined in (1)) as the unit of flow-rate. The dimensionless rate of inflow q due to precipitation will be assumed constant and uniform.

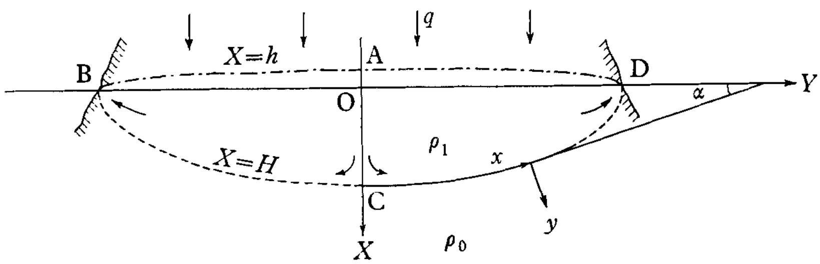

FIGURE 6. Idealized two-dimensional Ghyben–Herzberg lens formed by ground water overlying saline water. – – – –, Mixing layer; –·–·–·–, free surface of lens.

The equations of continuity and motion can be written as

$$qY = \int_h^H V\,dX, \tag{27}$$

$$\partial p/\partial X + U = -1, \quad \partial p/\partial Y + V = 0, \tag{28}$$

where $H(Y)$ and $h(Y)$ are the X co-ordinates of the mixing layer and free surface, and p is defined as in §2 with L replaced by L_1. The potential-flow problem is to be solved subject to the boundary conditions $p = 0$ on $X = H$ and

$$p = -h\rho_0/|\rho_1 - \rho_0| \quad \text{on} \quad X = h,$$

and the condition $H = h = 0$ at $Y = 1$. Then an approximate solution, in terms of an expansion valid as $q \to 0$, is found to be

$$\left.\begin{aligned} p &= H - X + O(q^{\frac{3}{2}}), \\ V &= -\partial H/\partial Y + O(q^{\frac{3}{2}}), \\ -h/H &= (|\rho_1 - \rho_0|/\rho_1)\{1 + O(q)\}, \\ H^2 &= q_1(1 - Y^2) + O(q^2), \end{aligned}\right\} \tag{29}$$

where $q_1 = (\rho_1/\rho_0)q \approx q$, and where p, V, X and H are $O(q^{\frac{1}{2}})$ while $U = O(q)$. The result (29) is usually derived by means of a physical argument known as Dupuit–Forchheimer theory (Scheidegger 1957, p. 80).

At points not far from D in figure 6, $H \propto (1 - Y)^{\frac{1}{2}}$ approximately, which agrees with the observation that the depth of the lens is nearly proportional to the one-half power of the distance from shore (Carrier 1958).

The solution (29) can be used to calculate the mixing-layer thickness along the curve CD, excluding the neighbourhood of the singularity at D where the approximate theory breaks down. In place of the boundary condition (10*b*), expression (29) gives

$$u(x, -\infty) = V \sec\alpha = \sin\alpha\{1+O(\alpha^2)\}$$

which is therefore valid for α small.

Now, the formula (15) for the mixing-layer thickness and the definition (3) of the Rayleigh number are based upon a length scale L equal to the radius of curvature at C; in terms of the length scale $L_1 = \text{OD}$ that radius is equal to $q^{-\frac{1}{2}}$. It is convenient to retain the original length L in defining the variable x in the intrinsic equation, which is obtained from (29) in the parametric form

$$\left.\begin{aligned} \tan\alpha &= -dH/dY = q^{\frac{1}{2}}Y/(1-Y^2)^{\frac{1}{2}} + O(q^{\frac{3}{2}}), \\ x/q^{\frac{1}{2}} &= \int_0^Y \sec\alpha\, dY = Y + \tfrac{1}{2}q(\tanh^{-1}Y - Y) + O(q^2). \end{aligned}\right\} \tag{30}$$

Substitution of (30) into (15) gives the dimensionless mixing-layer thickness according to first-order theory as

$$h^{\frac{1}{2}}/\sin\alpha = Y^{-1}(1-Y^2)^{\frac{1}{2}}\{1-(1-Y^2)^{\frac{1}{2}}\}^{\frac{1}{2}} + O(q). \tag{31}$$

This remains finite as $q \to 0$, and tends to the finite limit of $2^{-\frac{1}{2}} + O(Y)^2$ in the neighbourhood of the stagnation point at C, as in the case of the stagnation flow described in §4. Hence the actual mixing-layer thickness varies as the square root of the radius of curvature of the Ghyben–Herzberg lens. Expression (31) also shows that the mixing-layer thickness becomes gradually thinner with increasing Y, but fails to represent the thickness accurately as the coast D is approached. In that region it would be necessary to use the exact equations (27) and (28) to obtain the potential-flow solution, and to take into account the existence of a seepage line at the coast itself.

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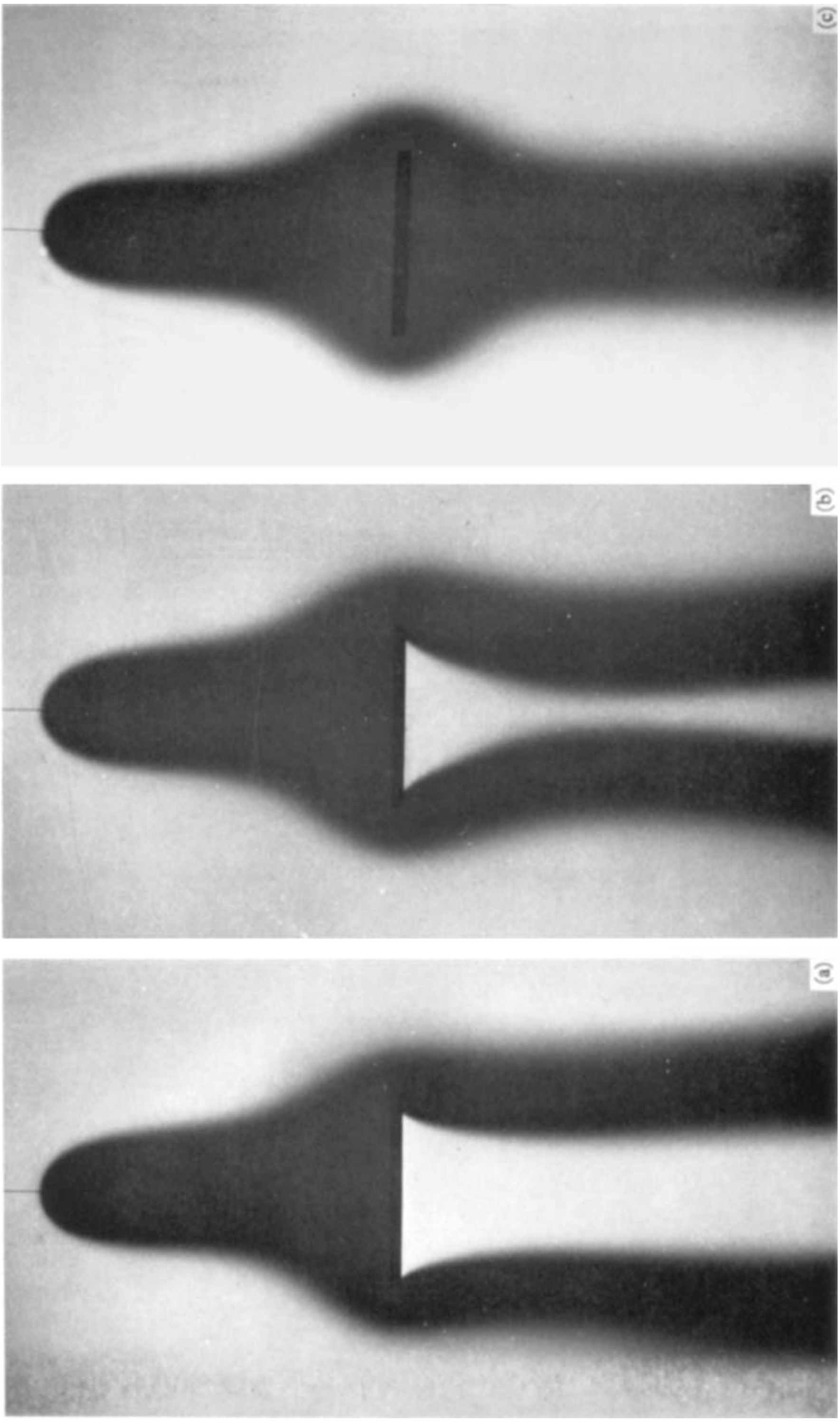

FIGURE 4. Development of steady flow about an impermeable horizontal strip of length 5 cm in a Hele-Shaw cell. These photographs correspond to elapsed times of (*a*) 2 hr, (*b*) 5 hr and (*c*) 48 hr from the time at which the flow first passed the obstacle. The parameters of the steady flow shown in (*c*) are approximately $a = 2{\cdot}65$, $X_0 = 10$ and $\lambda = 1000$. (These are defined in §5.)